Definitions
Geometry qualifying course
MSU, Fall 2016

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This document was made as a way to study the material from the fall semester differential geometry qualifying course at Michigan State University, in fall of 2016. It serves as a companion document to the "Theorems" review sheet for the same class. The main textbook for the course was *Introduction to Smooth Manifolds* by John Lee, and this document closely follows the order of material in that book.

Contents

0 Appendix A: Topology

Definition 0.1. *A topological space X is* ***Hausdorff*** *if for every $x, y \in X$ there exist open sets U, V with$x \in U, y \in V$ and $U \cap V = \emptyset$.*

Definition 0.2. *A* ***basis*** *for a topological space X is a collection $\mathcal{B}$ such that any open set $U \subset X$ can be written as a union $U = \bigcup_{\alpha \in A} B_\alpha$ where each $B_\alpha \in \mathcal{B}$.*

Definition 0.3. *A topological space is* ***second-countable*** *if it has a countable basis.*

Definition 0.4. *Let X be a topological space. A subset $A \subset X$ is* ***precompact*** *if the closure of A is compact.*

Definition 0.5. *A topological space X is* ***locally compact*** *if every point x is contained in an open set U such that U is a subset of a compact set.*

Definition 0.6. *Let X be a topological space. A collection of subset $\{A_\alpha\}$ is* ***locally finite*** *if every point in X has an (open) neighborhood U such that $U \cap A_\alpha \neq \emptyset$ for only finitely many α.*

Definition 0.7. *Let X be a topological space and $\mathcal{U}$ be an open cover. A* ***refinement*** *of $\mathcal{U}$ is another cover $\mathcal{V}$ so that each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.*

Definition 0.8. *A topological space X is* ***paracompact*** *if every open cover admits an open, locally finite refinement.*

1 Chapter 1 - Defining manifolds

Definition 1.1. *A* ***topological n-manifold*** *is a topological space that is Hausdorff, second-countable, and locally Euclidean of dimension n.*

Definition 1.2. *For a topological n-manifold M, a* ***coordinate chart*** *is a pair (U, ϕ) where U is an open subset of M and $\phi : U \to \hat{U}$ is a homeomorphism (where $\hat{U} \subset \mathbb{R}^n$).*

Definition 1.3. *Let M be a topological n-manifold and let (U, ϕ) be a coordinate chart. Then $\phi(p) = (x^1(p), x^2(p), \ldots, x^n(p))$ for some functions $x^i : U \to \mathbb{R}^n$. The functions x^i are called* ***local coordinates*** *or* ***coordinate functions*** *on U.*

Definition 1.4. *Let $M_1, \ldots, M_k$ be topological manifolds of dimension $n_1, \ldots, n_k$ respectively. The* ***product manifold*** *is the cartesian product $\prod_i M_i$ endowed with the product topology.*

Definition 1.5. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. A function $F : U \to V$ can be written as*

$$F(x_1, \ldots, x_n) = (F^1(x_1, \ldots, x_n), \ldots, F^m(x_1, \ldots, x_n))$$

F is ***smooth*** *if each F^i has continuous partial derivatives of all orders.*

Definition 1.6. *Let U, V be open subsets of Euclidean spaces. A map $F : U \to V$ is a* ***diffeomorphism*** *if F is bijective, smooth, and has a smooth inverse.*

Definition 1.7. *Let M be a topological n-manifold. Two charts $(U, \phi), (V, \psi)$ are* ***smoothly compatible*** *if the map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.*

Definition 1.8. *Let M be a topological n-manifold. An* ***atlas*** *for M is a collection of smooth charts whose domains cover M.*

Definition 1.9. *Let M be a topological n-manifold. A* ***smooth atlas*** *for M is an atlas consisting of smoothly compatible charts.*

Definition 1.10. *Let M be a topological n-manifold. A* ***maximal smooth atlas*** *for M is a smooth atlas that is not contained in any other smooth atlas for M.*

Definition 1.11. *Let M be a topological n-manifold. A* ***smooth structure*** *for M is a maximal smooth atlas for M.*

Definition 1.12. *A* ***smooth manifold*** *is a topological n-manifold with a smooth structure.*

Definition 1.13. *Let M be a smooth manifold. Then any chart (U, ϕ) contained in the given maximal smooth atlas is called a* ***smooth chart*** *or a* ***smooth coordinate chart****.*

Definition 1.14. *A* ***smooth coordinate ball*** *is a smooth coordinate chart whose domain is homeomorphic to a ball in Euclidean space.*

Definition 1.15. *Let M be a smooth manifold. A* ***regular coordinate ball*** *is a smooth coordinate ball whose closure is contained in another smooth coordinate ball in a nice way. More precisely, (B, ϕ) is a regular coordinate ball if there is another smooth chart (B', ϕ') such that $\overline{B} \subset B'$ and*

$$\phi'(B) = B(0, r)$$
$$\phi'(\overline{B}) = \overline{B}(0, r)$$
$$\phi'(B') = B(0, r')$$

for some $0 < r < r'$.

Definition 1.16. *Let V be an n-dimensional real vector space, endowed with the topology so that scalar multiplication and vector addition are continuous. Choose an ordered basis $\{E_1, \dots E_n\}$. Then define $E : \mathbb{R}^n \to V$ by*

$$E(x^1, \dots, x^n) = \sum_{i=1}^{n} x^i E_i$$

Then E is a homeomorphism, so (V, E^{-1}) is a chart. This single chart induces a maximal atlas on V. (One can check that this chart is smoothly compatible by an analogous chart induced by another choice of basis for V.) This is called the ***standard smooth structure on the vector space V****.*

Definition 1.17. *Let M be a smooth manifold. A* ***open submanifold*** *of M is any open subset U. The smooth charts for U are of the form (V, ψ) where $V \subset U$.*

Definition 1.18. *A* ***closed manifold*** *is a manifold that is compact and has empty boundary.*

Definition 1.19. *Let $U \subset \mathbb{R}^n$ and $\phi : U \to \mathbb{R}$ be a smooth function. The set $\phi^{-1}(c)$ is a* ***level set*** *of ϕ.*

2 Chapter 2 - Smooth functions

Definition 2.1. *Let M be a smooth n-manifold, and $f : M \to \mathbb{R}^k$. f is a* ***smooth function*** *if for every $p \in M$, there exists a smooth chart (U, ϕ) where $p \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth (has continuous partial derivatives of all orders).*

Definition 2.2. *Let M be a manifold. Then we define $\boldsymbol{C^\infty(M)} = \{f : M \to \mathbb{R}\}$ where f is smooth.*

Definition 2.3. *Let M be a smooth manifold. We define addition and multiplication on $C^\infty(M)$ pointwise, as well as scalar multiplication from $\mathbb{R}$. For $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,*

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x) \qquad (af)(x) = a(f(x))$$

This gives $C^\infty(M)$ the structure of a commutative and associative algebra over $\mathbb{R}$.

Definition 2.4. *Let M be a smooth manifold, $f : M \to \mathbb{R}^k$, and let (U, ϕ) be a chart for M. The* ***coordinate representation of $\boldsymbol{f}$*** *is the map $\widehat{f} : \phi(U) \to \mathbb{R}^k$ defined by $\widehat{f} = f \circ \phi^{-1}$.*

Definition 2.5. *Let M, N be smooth manifolds. A map $F : M \to N$ is* ***smooth*** *if for every $p \in M$ there exist smooth charts $(U, \phi), (V, \psi)$ with $p \in U \subset M$ and $F(U) \subset V \subset N$ such that $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth (in the calculus sense).*

Definition 2.6. *Let $F : M \to N$ be a smooth map. For any charts (U, ϕ) for M and (V, ψ) for N, the* ***coordinate representation of $\boldsymbol{F}$*** *is the map $\widehat{\boldsymbol{F}} = \psi \circ F \circ \phi^{-1}$.*

Definition 2.7. *A* ***diffeomorphism*** *is a smooth map $F : M \to N$ that is a bijection and has a smooth inverse. If there is a diffeomorphism between two manifolds, they are* ***diffeomorphic****.*

Definition 2.8. *Let $f : M \to \mathbb{R}^k$. The* ***support of $\boldsymbol{f}$*** *is the closure of $\{p \in M : f(p) \neq 0\}$.*

Definition 2.9. *Let $f : M \to \mathbb{R}^k$. If $\operatorname{supp} f \subset U$ for some $U \subset M$, then f is* ***supported in $\boldsymbol{U}$****.*

Definition 2.10. *Let $f : M \to \mathbb{R}^k$. If $\operatorname{supp} f$ is compact, then f is* ***compactly supported****.*

Definition 2.11. *Let M be a topological space and let $\{U_\alpha\}_{\alpha \in A}$ be a collection of subsets. The collection $\{U_\alpha\}$ is* ***locally finite*** *if for each $p \in M$, there is a neighborhood V such that $V \cap U_\alpha \neq \emptyset$ for only finitely many U_α.*

Definition 2.12. *Let M be a topological space and let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an open cover of M. A* ***partition of unity subordinate to $\boldsymbol{\mathcal{X}}$*** *is a family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ such that*

1. *$0 \leq \psi_\alpha(p) \leq 1$ for all $\alpha \in A, p \in M$.*
2. *$\operatorname{supp} \psi_\alpha \subset X_\alpha$ for all $\alpha \in A$.*
3. *The collection $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$ is locally finite.*

4. $\sum_{\alpha \in A} \psi_\alpha(p) = 1$ *for all* $p \in M$.

(Note that because the supports of ψ_α *is a locally finite collection, the sum over* $\alpha \in A$ *has only finitely many nonzero terms, so there are no issues of convergence.)*

Definition 2.13. *Let* M *be a topological space. A* ***smooth partition of unity*** *for* M *is a partition of unity in which each of the functions* ψ_α *is smooth.*

Definition 2.14. *Let* M *be a topological space,* $A \subset M$ *be closed, and* $U \subset M$ *closed, with* $A \subset U$. *A* ***bump function for A supported in U*** *is a continuous function* $\psi : M \to \mathbb{R}$ *with* $0 \leq \psi \leq 1$ *on* M, $\psi = 1$ *on* A, *and* $\operatorname{supp} \psi \subset U$.

Definition 2.15. *Let* M, N *be smooth manifolds and* $A \subset M$. *We say that* $F : A \to N$ *is* ***smooth on A*** *if for every* $p \in A$, *there is an open subset* W *with* $p \in W \subset M$ *and a smooth map* $\widetilde{F} : W \to N$ *such that* $\widetilde{F}|_{W \cap A}(x) = F(x)$ *for* $x \in W \cap A$.

Definition 2.16. *Let* M *be a topological space. A* ***exhaustion function for M*** *is a continuous function* $f : M \to \mathbb{R}$ *such that* $f^{-1}((-\infty, a])$ *is compact for every* $a \in \mathbb{R}$.

3 Chapter 3 - Tangent bundle

Recall that a linear map $v : C^\infty(M) \to \mathbb{R}$ has the following properties: For $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

$$v(f + g) = vf + vg \qquad v(af) = av(f)$$

Definition 3.1. *Let* M *be a smooth manifold and* $p \in M$. *A* ***derivation at p*** *is a linear map* $v : C^\infty(M) \to \mathbb{R}$ *that satisfies*

$$v(fg) = f(p) \cdot vg + g(p) \cdot vf$$

for all $f, g \in C^\infty(M)$.

Definition 3.2. *Let* M *be a manifold and* $p \in M$. *The set of derivations at* p *is the* ***tangent space to M at p****, and it is denoted* $\boldsymbol{T_pM}$. *An element* $v \in T_pM$ *is a* ***tangent vector at p****.*

Definition 3.3. *Let* M, N *be smooth manifolds and* $F : M \to N$ *be smooth. The* ***pullback of F*** *is the map* $F^* : C^\infty(N) \to C^\infty(M)$ *defined by* $F^*(f) = f \circ F$.

Definition 3.4. *Let* M, N *be smooth manifolds and* $F : M \to N$ *be a smooth map and let* $p \in M$. *The* ***differential of F at p****, also called the* ***pushforward of F at p*** *is a map* $dF_p : T_pM \to T_{F(p)}N$. *It maps a derivation at* p *to a derivation at* $F(p)$, *so for* $v \in T_pM$, $dF_p(v)$ *acts on functions* $f \in C^\infty(N)$.

$$dF_p : T_pM \to T_{F(p)}N$$
$$dF_p(v)(f) = v(f \circ F)$$

Another notation that is used for the differential of F at p is

$$F_* : T_pM \to T_{F(p)}N$$
$$F_*(v)(f) = v(f \circ F) = vF^*(f)$$

As a helpful reminder, note that

$$\begin{array}{ll} f \in C^\infty(N) & f : N \to \mathbb{R} \\ v \in T_pM & v : C^\infty(M) \to \mathbb{R} \\ dF_p(v) \in T_{F(p)}N & dF_p(v) : C^\infty(N) \to \mathbb{R} \end{array}$$

Definition 3.5. *Let M be a smooth n-manifold, $p \in M$, and (U, ϕ) a smooth chart with $p \in U$. Let $\widehat{U} = \phi(U)$. Since $\phi : U \to \widehat{U}$ is a diffeomorphism, $\phi_* : T_pU \to T_{\phi(p)}\widehat{U}$ is an isomorphism, so we have a basis $\left\{ \frac{\partial}{\partial x^i}\big|_{\phi(p)} \right\}_{i=1}^n$ for $T_{\phi(p)}\widehat{U}$. Then we define the* ***coordinate vectors at p*** *(associated with the coordinates (U, ϕ)) to be*

$$\boldsymbol{\frac{\partial}{\partial x^i}\bigg|_p} = (\phi_*)^{-1} \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right)$$

Then we see that

$$\frac{\partial}{\partial x^i}\bigg|_p f = (\phi_*)^{-1} \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}} \right) f = \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}} \right) (\phi^{-1})^* f = \left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}} \right) (f \circ \phi^{-1}) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p})$$

where $\widehat{f} = f \circ \phi^{-1}$ and $\widehat{p} = \phi(p)$. The set

$$\left\{ \frac{\partial}{\partial x^i}\bigg|_p \right\}_{i=1}^n$$

is a ***coordinate basis*** *for T_pM.*

Definition 3.6. *Let M be a smooth manifold. The* ***tangent bundle of M*** *is the disjoint union of all tangent spaces,*

$$\boldsymbol{TM} = \bigsqcup_{p \in M} T_pM$$

Definition 3.7. *Let M be a smooth manifold and TM the tangent bundle. The* ***natural projection*** *is the map $\pi : TM \to M$ defined by $(p, v) \mapsto p$.*

Definition 3.8. *Let M be a smooth manifold and TM the tangent bundle. Let $\pi : TM \to M$ be the natural projection. For a chart (U, ϕ) for M, let $\phi(p) = (x^1(p), \ldots x^n(p))$, and define $\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by*

$$\widetilde{\phi} \left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = (x^1(p), \ldots, x^n(p), v^1, \ldots v^n)$$

Then the collection of charts of the form $(\pi^{-1}(U), \widetilde{\phi})$ are smooth charts for TM. These are called the ***natural coordinates on TM****.*

Definition 3.9. *Let M, N be smooth manifolds and let $F : M \to N$ be smooth. The* ***global differential*** *is a map $dF : TM \to TN$. For $p \in M, v \in T_pM$, we have $(p, v) \in TM$, and dF is the map*

$$(p, v) \mapsto (F(p), dF_p(v))$$

Notice that $dF_p(v) \in T_{F(p)}N$, so $(F(p), dF_p(v)) \in TN$.

Definition 3.10. *Let M be a smooth manifold. A* ***curve in M*** *is a continuous map $\gamma : J \to M$ where $J \subset \mathbb{R}$ is an interval.*

Definition 3.11. *Let M be a smooth manifold and let $\gamma : J \to M$ be a curve in M. For $t_0 \in J$, the* ***velocity of γ at t_0*** *is a particular vector in $T_{\gamma(t_0)}M$. It is denoted $\boldsymbol{\gamma'(t_0)}$, and is given by*

$$\gamma'(t_0) = \left.\frac{d\gamma}{dt}\right|_{t_0} \in T_{\gamma(t_0)}M$$

If (U, x^i) is a coordinate chart containing $\gamma(t_0)$, then

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}$$

4 Chapter 4 - Submersions and immersions

Definition 4.1. *A map $F : X \to Y$ of topological spaces is* ***proper*** *if the preimage of any compact set is compact.*

Definition 4.2. *Let $F : M \to N$ be a smooth map. For $p \in M$, the* ***rank of F at p*** *is the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.*

Definition 4.3. *Let $F : M \to N$ be a smooth map. F has* ***constant rank*** *if the rank of F at p is the same for all $p \in M$. If F has constant rank, we call the* ***rank of F****.*

Definition 4.4. *Let $F : M \to N$ be a smooth map. If the rank of F at p is $\min(\dim M, \dim N)$ then F* ***has full rank at p****.*

Definition 4.5. *Let $F : M \to N$ be a smooth map. F* ***has full rank*** *if F has rull rank at every $p \in M$.*

Definition 4.6. *A* ***smooth submersion*** *is a smooth map $F : M \to N$ with $\operatorname{rank} F = \dim N$ (this means that $dF_p = F_*$ is surjective for all $p \in M$).*

Definition 4.7. *A* ***smooth immersion*** *is a smooth map $F : M \to N$ with $\operatorname{rank} F = \dim M$ (this means that $dF_p = F_*$ is injective for all $p \in M$).*

Definition 4.8. *Let $F : M \to N$ be a smooth map. F is a* ***local diffeomorphism*** *if for every $p \in M$, there exists a neighborhood U of p such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.*

Definition 4.9. *Let M, N be smooth manifolds. A* ***smooth embedding of M into N*** *is a smooth immersion $F : M \to N$ that is also a homeomorphism onto its image. That is, a smooth embedding is injective, continuous, and has a continuous inverse.*

Definition 4.10. *Let X, Y be topological spaces and $F : X \to Y$ a continuous map. F is a* ***topological embedding*** *if F is injective and is a homeomorphism onto its image.*

Definition 4.11. *Let X, Y be topological spaces and $F : X \to Y$ be a continuous map. F is a* ***topological immersion*** *if every $x \in X$ has a neighborhood U such that $F|_U : U \to Y$ is a topological embedding.*

Definition 4.12. *Let M, N be smooth manifolds and let $\pi : M \to N$ be a continuous map. A* ***section of π*** *is a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$.*

Definition 4.13. *Let M, N be smooth manifolds and let $\pi : M \to N$ be a continuous map. A* ***local section of π*** *is a continuous map $\sigma : U \to M$ defined on some open subset $U \subset N$ such that $\pi \circ \sigma = \mathrm{Id}_U$.*

Definition 4.14. *Let X, Y be topological spaces and $\pi : X \to Y$ a continuous map. Then π is a* ***topological submersion*** *if every $x \in X$ is in the image of a local section of π.*

5 Chapter 5 - Critical points of smooth functions

Definition 5.1 (repeated from Chapter 1 for convenience)**.** *Let M be a smooth manifold. A* ***open submanifold*** *of M is any open subset U. The smooth charts for U are of the form (V, ψ) where $V \subset U$.*

Definition 5.2. *Let M be a smooth manifold. An* ***embedded submanifold*** *is a subset $S \subset M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding. These may also be called* ***regular submanifolds****. M is called the* ***ambient manifold*** *for S.*

Definition 5.3. *Let S be an embedded submanifold in M. Then the* ***codimension of S in M*** *is $\dim M - \dim S$. If S has codimension 1, then S is called an* ***embedded hypersurface****.*

Definition 5.4. *If U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$ a* ***k-dimensional slice of U*** *is a subset of the form*

$$S = \{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n\}$$

for some real constants $c^{k+1}, \ldots, c^n$. A k-dimensional slice of U is also simply called a ***k-slice****.*

Definition 5.5. *Let M be a smooth n-manifold and let (U, ϕ) be a smooth chart. If $S \subset U$ such that $\phi(S)$ is a k-slice of $\phi(U)$, then we say S is a* ***k-slice*** *of U.*

Definition 5.6. *Let M be a smooth n-manifold. A subset $S \subset M$ satisfies the* ***local k-slice condition*** *if each point of S is contained in the domain of a smooth chart (U, ϕ) such that $S \cap U$ is a k-slice of U.*

Definition 5.7. *Let M, N be smooth manifolds. For $p \in N$, the subset $M \times \{p\}$ is a* ***slice*** *of $M \times N$.*

Definition 5.8. *Let M, N be smooth manifolds with dimension m, n respectively. Let $U \subset M$ be open and $f : U \to N$ be a smooth map. The* ***graph of f*** *is the set*

$$\Gamma(f) = \{(x, y) \in M \times N : x \in U, y = f(x)\}$$

Definition 5.9. *Let M be a smooth manifold and $S \subset M$ be an embedded submanifold. S is a* ***properly embedded submanifold*** *if the inclusion $S \hookrightarrow M$ is a proper map.*

Definition 5.10 (repeated from chapter 1 for convenience)**.** *Let $\phi : M \to N$ be a map and $c \in N$. The set $\phi^{-1}(c)$ is a* ***level set*** *of ϕ.*

Definition 5.11. *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth map. A point $p \in M$ is* ***regular point*** *of ϕ if $d\phi_p : T_pM \to T_{\phi(p)}N$ is surjective.*

Definition 5.12. *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth map. A point $p \in M$ is* ***critical point*** *of ϕ if it is not a regular point.*

Definition 5.13. *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth map. A point $c \in N$ is a* ***regular value of ϕ*** *if every point in the level set $\phi^{-1}(c)$ is a regular point.*

Definition 5.14. *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth map. A point $c \in N$ is a* ***critical value of ϕ*** *if there is some point in the level set $\phi^{-1}(c)$ is a critical point.*

Definition 5.15. *Let M, N be smooth manifolds and $\phi : M \to N$ be a smooth map. A level set $\phi^{-1}(c)$ is a* ***regular level set*** *if c is a regular value of ϕ.*

Definition 5.16. *Let M be a smooth manifold with or without boundary. An* ***immersed submanifold of M*** *is a subset $S \subset M$ endowed with a topology (not necessarily the subspace topology) such that S is a topological manifold, and S has a smooth structure such that the inclusion map $S \hookrightarrow M$ is a smooth immersion. The* ***codimension of S*** *is $\dim M - \dim S$.*

Definition 5.17. *The term* ***smooth submanifold*** *refers to an immersed submanifold.*

Definition 5.18. *Let M be a smooth manifold and S an immersed submanifold. S is* ***weakly embedded in M*** *if every smooth map $F : N \to M$ whose image lies in S is smooth as a map from N to S. These may also be called* ***initial submanifolds****.*

Definition 5.19. *Let M be a smooth manifold with boundary. For $p \in \partial M$, a vector $v \in T_pM \setminus T_p\partial M$ is* ***inward pointing*** *if for some $\epsilon > 0$ there exists a smooth curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. The vector v is* ***outward pointing*** *if there is a curve $\gamma : (-\epsilon, 0]$ with $\gamma(0) = p$ and $\gamma'(0) = v$.*

6 Chapter 6 - Sard's theoerm

Definition 6.1. *Let M be a smooth n-manifold with or without boundary. A subset $A \subset M$ has* ***measure zero in M*** *if for every smooth chart (U, ϕ) the subset $\phi(A \cap U)$ has measure zero in $\mathbb{R}^n$.*

7 Chapter 7 - Lie groups

Definition 7.1. *A **Lie group** is a smooth manifold without boundary that is also a group, such that the multiplication $m : G \times G \to G$ and inversion $i : G \to G$ given by $m(g, h) = gh$ and $i(g) = g^{-1}$ are both smooth maps.*

Definition 7.2. *Let G be Lie group. We define the **left translation** and **right translation** maps $L_g, R_g : G \to G$ by*

$$L_g(h) = gh \qquad R_g(h) = hg$$

Note that L_g is the composition of $h \mapsto (g, h)$ and $(g, h) \mapsto gh$ so L_g is smooth. Since $L_g \circ L_{g^{-1}} = \mathrm{Id}_G$, L_g is a diffeomorphism.

Definition 7.3. *A **Lie group homomorphism** is a smooth map $F : G \to H$ that is also a group homomorphism.*

Definition 7.4. *A **Lie group isomorphism** is a diffeomorphism $F : G \to H$ that is also a group homomorphism.*

Definition 7.5. *Let G be Lie group. A **Lie subgroup** is a subgroup that is also an immersed submanifold.*

Definition 7.6. *Let G be a group and S a subset. The **subgroup generated by S** is the set of all elements that can be expressed as finite products of elements of S and their inverses.*

Definition 7.7. *Let G be a Lie group and M, N be smooth manifolds with left G-actions. A map $F : M \to N$ is **equivariant** with respect to these actions if*

$$F(g \cdot p) = g \cdot F(p)$$

for $g \in G$ and $p \in M$. That is, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ {\scriptstyle \theta_g}\downarrow & & {\scriptstyle \phi_g}\downarrow \\ M & \xrightarrow{F} & N \end{array}$$

where $\theta_g : M \to M$ is the map $p \mapsto g \cdot p$ and $\phi_g : N \to N$ is the map $x \mapsto g \cdot x$.

Definition 7.8. *The **orthogonal group** is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ such that $A^T A = I_n$.*

8 Chapter 8 - Vector fields

Definition 8.1. *Let M be a smooth manifold. A **vector field** on M is a section of $\pi : TM \to M$. That is, a vector field is a continuous map $X : M \to TM$ denoted $X \mapsto X_p$ such that $X_p \in T_pM$ for every $p \in M$.*

Definition 8.2. *A **smooth vector field** is a vector field that is smooth as a map from TM to M.*

Definition 8.3. *A* ***rough vector field*** *is a map $X : M \to TM$ such that $X_p \in T_pM$ for every p. (Not necessarily a smooth or continuous map.)*

Definition 8.4. *If (U, x^i) is a smooth coordinate chart for M and X is a vector field on M, then the* ***component functions*** *of X are the functions $X^i : U \to \mathbb{R}$ such that*

$$X_p = \sum_i X^i(p) \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

Definition 8.5. *Let M be a smooth manifold. We denote the set of all smooth vector fields on M by* $\boldsymbol{\mathfrak{X}(M)}$*. We define addition in $\mathfrak{X}(M)$ by*

$$(X + Y)_p = X_p + Y_p$$

We define scalar multiplication from $\mathbb{R}$ by

$$(aX)_p = a(X_p)$$

Then $\mathfrak{X}(M)$ is a vector space over $\mathbb{R}$ with these operations. (Note that the sum of smooth vector fields is smooth.)

Definition 8.6. *We define an action $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $(f, X) \mapsto fX$ where*

$$(fX)_p = f(p)X_p$$

Note that if f and X are smooth then fX is smooth. This action makes $\mathfrak{X}(M)$ a $C^\infty(M)$ module.

Definition 8.7. *We define an action $\mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M)$ by $(X, f) \mapsto Xf$ where*

$$(Xf)(p) = X_p f$$

Because of this action, for each $X \in \mathfrak{X}(M)$, we have a map $C^\infty(M) \to C^\infty(M)$ given by $f \mapsto Xf$. This map is a derivation of $C^\infty(M)$. That is,

$$X(fg) = f(Xg) + g(Xf)$$

Definition 8.8. *Let M be a smooth manifold with $A \subset M$. An ordered k-tuple $(X_1, \ldots, X_k)$ of vector fields on A is* ***linearly independent*** *if $(X_1|_p, \ldots, X_k|_p)$ is linearly independent for each $p \in A$.*

Definition 8.9. *Let M be a smooth n-manifold. A* ***local frame for M*** *is a linearly independent ordered n-tuple of vector fields defined on an open subset U. Then their values at any p form a basis for T_pM.*

Definition 8.10. *A* ***global frame*** *for M is a local frame defined on all of M.*

Definition 8.11. *A smooth manifold is* ***parallelizable*** *if it has a smooth global frame.*

Definition 8.12. *Let $F : M \to N$ be a smooth map, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. X and Y are **F-related** if for each $p \in M$, we have $dF_p(X_p) = Y_{F(p)}$. (In other notation, $F_*(X_p) = Y_{F(p)}$. In other words, the following diagram commutes.*

$$\begin{array}{ccc} M & \xrightarrow{X} & TM \\ {\scriptstyle F}\downarrow & & {\scriptstyle F_*}\downarrow \\ N & \xrightarrow{Y} & TN \end{array}$$

Definition 8.13. *If $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$, the **pushforward** of X by F is the vector field F_*X defined by*

$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$$

Definition 8.14. *Let $X, Y \in \mathfrak{X}(M)$. The **Lie bracket** of X and Y is the operator $[X, Y] : C^\infty(M) \to C^\infty(M)$ given by*

$$[X, Y]f = XYf - YXf$$

Note that $[X, Y] \in \mathfrak{X}(M)$. Pointwise, we have

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf)$$

Definition 8.15. *Let G be a Lie group. A vector field X on G is **left-invariant** if*

$$(L_g)_*X = X$$

for all $g, h \in G$.

Definition 8.16. *Let G be a Lie group. The **Lie algebra** of G, denoted $\mathrm{Lie}(G)$, is the vector space of smooth left-invariant vector fields, with the usual bracket.*

9 Chapter 9 - Flows

Definition 9.1. *Let V be a vector field on M. An **integral curve of V** is a smooth curve $\gamma : J \to M$ such that $\gamma'(t) = V_{\gamma(t)}$ for $t \in J$.*

Definition 9.2. *Let M be a smooth manifold. A **global flow** on M is a map $\theta : \mathbb{R} \times M \to M$ such that*

$$\theta(t, \theta(s, p)) = \theta(t + s, p) \qquad \theta(0, p) = p$$

Given a global flow θ, we define a one-parameter family of maps $\theta_t : M \to M$ defined by $\theta_t(p) = \theta(t, p)$. This family satisfies $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ and $\theta_0 = \mathrm{Id}_M$. Note that θ_t is a diffeomorphism. We also have a family of maps $\theta^p(t) : \mathbb{R} \to M$ given by $\theta^p(t) = \theta(t, p)$. The image of θ^p is the orbit of p.

Definition 9.3. *Let M be a smooth manifold and $\theta : \mathbb{R} \times M \to M$ a global flow. The **infinitesimal generator** of θ is the rough vector field V defined by*

$$V_p = \frac{d}{dt}\theta^p(t)\bigg|_{t=0}$$

Definition 9.4. *Let M be a smooth manifold. A* ***flow domain*** *is a subset D of $\mathbb{R} \times M$ such that for each $p \in M$, the set $D^p = \{t \in \mathbb{R} : (t, p \in D\}$ is an open interval containing 0.*

Definition 9.5. *Let M be a smooth manifold. A* ***flow*** *on M is a map $\theta : D \to M$ where D is a flow domain, and θ satisfies*

$$\theta(0, p) = p \qquad \theta(t, \theta(s, p)) = \theta(s + t, p)$$

for $s \in D^p$ and $t \in D^{\theta(s,p)}$ such that $s + t \in D^p$.

Definition 9.6. *A* ***maximal integral curve*** *is an integral curve $\gamma : J \to M$ such that γ cannot be (smoothly) extended to any interval larger than J.*

Definition 9.7. *A* ***maximal flow*** *on a manifod is a flow on M that cannot be (smoothly) extended to any larger flow domain.*

Definition 9.8. *Let θ be a flow on M with flow domain D. We define the set*

$$M_t = \{p \in M : (t, p) \in D\}$$

Definition 9.9. *A vector field is* ***complete*** *if it generates a global flow. That is, all of the integral curves are defined for all $t \in \mathbb{R}$.*

Definition 9.10. *Let V be a vector field. A* ***singular point*** *is a point p such that $V_p = 0$. If $V_p \neq 0$, then p is a* ***regular point****.*

Definition 9.11. *Let M be a manifold and $V, W \in \mathfrak{X}(M)$. V and W* ***commute*** *if $VWf = WVf$ for every $f \in C^\infty(M)$. Equivalently, V, W commute if $[V, W] = 0$.*

Definition 9.12. *Let M be a manifold and θ a flow on M. A vector field W is* ***invariant under the flow of θ*** *if W is θ_t-related to itself for each t.*

10 Chapter 10 - Vector bundles

Definition 10.1. *Let $F : M \to N$ be a map. A* ***fiber*** *of F is the preimage of a single point, $F^{-1}(p)$.*

Definition 10.2. *Let $\pi : E \to M$ be a smooth map. A* ***local trivialization of E over U*** *is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that*

$$\pi_U \circ \phi = \pi$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times \mathbb{R}^k \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_U} & \\ U & & \end{array}$$

where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection. Also, for each $q \in U$, the restriction $\phi|_{E_q} : E_q \to \{q\} \times R^k$ is a vector space isomorphism.

Definition 10.3. *Let M be a smooth manifold. A **vector bundle of rank k over M** is a smooth manifold E with a surjective smooth map $\pi : E \to M$ such that each fiber $E_p = \pi^{-1}(p)$ is a k-dimensional (real) vector space, and for each $p \in M$, there is a chart U and a local trivialization of E over U.*

Definition 10.4. *Let $\pi : E \to M$ be a vector bundle. A **local section** of E is a continuous map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$, where $U \subset M$. A **global section** is a local section where $U = M$.*

Definition 10.5. *Let $\pi : E \to M$ and $\pi' : E' \to M'$ be smooth vector bundles. A **bundle homomorphism** is a smooth map $F : E \to E'$ such that there exists a smooth map $f : M \to M'$ such that $F_{E_p} : E_p \to E'_p$ is a linear map and the following diagram commutes.*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi'}\big\downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

*F is a **bundle isomorphism** if it is bijective and its inverse is a bundle homomorphism.*

Definition 10.6. *If $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles over M, a **bundle homomorphism** over M is a map $F : E \to E'$ such that $F_{E_p} : E_p \to E'_p$ is linear and $\pi' \circ F = \pi$.*

11 Chapter 11 - Differential 1-forms

Definition 11.1. *Let V be a finite-dimensional real vector space. A **covector** on V is a linear map $\omega : V \to \mathbb{R}$.*

Definition 11.2. *Let V be a finite-dimensional real vector space. The space of all covectors on V form a real vector space under pointwise addition and scalar multiplication of maps,*

$$(\omega + \alpha)(v) = \omega(v) + \alpha(v) \qquad (a\omega)(v) = a(\omega(v))$$

*This vector space is called the **dual space** of V and is denoted V^*.*

Definition 11.3. *Let M be a smooth manifold. The **cotangent space** at p is the dual of the tangent space, that is, $(T_pM)^*$. It is denoted T_p^*M. Elements of T_p^*M are maps from $T_pM \to \mathbb{R}$ and are called **covectors at p**.*

Definition 11.4. *The **cotangent bundle** of a manifold M is the disjoint union*

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

It is a vector bundle over M, of rank equal to $\dim M$.

Definition 11.5. *A section of the cotangent bundle is called a* ***covector field*** *or a* ***differential 1-form*** *or a* ***1-form****. More concretely, a 1-form is a map* $\omega : M \to T^*M$ *such that* $\omega_p \in T_p^*M$ *for every* p *(so* $\omega_p : T_pM \to \mathbb{R}$*). If* (U, x^i) *are local coordinates on* M*, a 1-form* ω *can be written as* $\omega = \omega_i dx^i$ *for some smooth functions* $\omega_i : U \to \mathbb{R}$*.*

Definition 11.6. *Let* $f \in C^\infty(M)$*. The* ***differential of f*** *is a 1-form denoted by* df*. That is, for* $p \in M$*,* df_p *is a map from* T_pM *to* $\mathbb{R}$*, given by*

$$df_p(v) = vf$$

Recall that $v \in T_pM$ *means that* v *is a linear map* $v : C^\infty(M) \to \mathbb{R}$ *that is a derivation at* p*.*

Definition 11.7. *Let* $f : V \to W$ *be a linear map between vector spaces. The* ***dual map****, denoted* f^**, is the map* $f^* : W^* \to V^*$ *defined by* $w \mapsto (v \mapsto w(f(v)))$*. That is,* $(f^*(w))(v) = w(f(v))$*.*

Definition 11.8. *Let* $F : M \to N$ *be a smooth map and let* $p \in M$*. The* ***pullback of F at p*** *is the dual map of the differential* $dF_p : T_pM \to T_{F(p)}N$*. That is,*

$$dF_p^*(w)(v) = w(dF_p(v))$$

Definition 11.9. *Let* $F : M \to N$ *be a smooth map and* ω *be a 1-form on* N*. The* ***pullback of ω by F*** *is the 1-form* $F^*\omega$*, which is defined by*

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)})$$

That is, $F^*\omega$ *is a map* $M \to T^*M$*, satisfing the above. Thus* $(F^*\omega)_p$ *is a map* $T_pM \to \mathbb{R}$*, given by*

$$(F^*\omega)_p(v) = dF_p^*(\omega_{F(p)}(v) = \omega_{F(p)} dF_p(v)$$

12 Chapter 12 - Differential k-forms

Definition 12.1. *Let* $V_1, \ldots, V_k, W$ *be vector spaces. The space* $\boldsymbol{L(V_1, \ldots, V_k; W)}$ *is the space of multilinear functions from* $V_1 \times \ldots \times V_k$ *to* W*.*

Definition 12.2. *Let* $V_1, \ldots, V_k, W_1, \ldots, W_l$ *be real vector spaces, and let* $F \in L(V_1, \ldots, V_k; \mathbb{R})$ *and* $G \in L(W_1, \ldots, W_l; \mathbb{R})$*. Then the* ***tensor product*** *of* F *and* G *is the map*

$$F \otimes G : V_1 \times \ldots \times V_k \times W_1 \times \ldots \times W_l \to \mathbb{R}$$

defined by

$$F \otimes G(v_1, \ldots, v_k, w_1, \ldots, w_l) = F(v_1, \ldots, v_k)G(w_1, \ldots, w_l)$$

In particular, if $\omega, \eta \in V^*$*, then*

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2)$$

Note that the tensor product is associative.

Definition 12.3. *Let V be a finite-dimensional real vector space. A* ***k-tensor*** *on V is a multilinear function from a k-fold product of V to $\mathbb{R}$, that is, $\alpha : V \times \ldots \times V \to \mathbb{R}$. The* ***rank*** *of α is k.*

Definition 12.4. *Let V be a vector space. The space of multilinear functions $\alpha : \prod_{i=1}^{k} V^* \to \mathbb{R}$ is called $\boldsymbol{T^k(V)}$.*

Definition 12.5. *A k-tensor is* ***symmetric*** *if it remains unchanged by interchanging pairs of arguments.*

Definition 12.6. *A k-tensor is* ***alternating*** *if it changes sign when two arguments are interchanged.*

Definition 12.7. *An alternating k-tensor is a* ***exterior form*** *of* ***k-covector****.*

Definition 12.8. *A* ***k-tensor field*** *or* ***k-form*** *on a manifold M is a map that assigns to each $p \in M$ an alternating k-linear function on T_pM. That is, if ω is a k-form, then ω_p is an alternating k-tensor on T_pM.*

Definition 12.9. *Let M be a smooth manifold. We define $\boldsymbol{\mathcal{T}(M)}$ to be the space of covariant k-tensor fields on M.*

13 Chapter 14 - Exterior derivative

Definition 13.1. *Let $k \in \mathbb{N}$. A* ***multi-index*** *of length k is an ordered k-tuple $I = (i_1, \ldots, i_k)$ where $i_1, \ldots, i_k \in \mathbb{N}$.*

Definition 13.2. *Let I be a multi-index of length k, and let $\sigma \in S_k$. Then we define $\boldsymbol{I_\sigma} = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$. Note that $I_{\sigma\tau} = (I_\sigma)_\tau$.*

Definition 13.3. *Let V be an n-dimensional vector space and $(\epsilon^1, \ldots, \epsilon^n)$ a basis for V^*. Let $I = (i_1, \ldots, i_k)$. We define a covariant k-tensor ϵ^I by*

$$\epsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \ldots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \ldots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

ϵ^I is a ***elementary alternating tensor*** *or* ***elementary k-covector****.*

Definition 13.4. *Let I, J be multi-indices of length k. We define a generalized Kronecker delta function δ^I_J by*

$$\delta^I_J = \begin{cases} \operatorname{sgn} \sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = I_\sigma \text{ for some permutation } \sigma \\ 0 & \text{otherwise} \end{cases}$$

Definition 13.5. *A multi-index $(i_1, \ldots, i_k)$ is* ***increasing*** *if $i_1 < \ldots < i_k$.*

Definition 13.6. *A summation over increasing multi-indices is denoted by a prime ′ symbol, as in* $\sum_I'$.

Definition 13.7. *The* ***alternation operator*** *is a map* $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ *defined by*

$$(\mathrm{Alt}\,\alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\,\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Definition 13.8. *Let* V *be a finite-dimensional vector space. Let* $\omega \in \Lambda^k(V)$ *and* $\eta \in \Lambda^l(V)$ *we define the* ***wedge product*** *by*

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta)$$

Definition 13.9. *Let* $I = (i_1, \ldots, i_k)$ *and* $J = (j_1, \ldots, j_l)$ *be multi-indices of length* k *and* l *respectively. Then we define*

$$\boldsymbol{IJ} = (i_1, \ldots, i_k, j_1, \ldots, j_l)$$

Definition 13.10. *Let* V *be an* n*-dimensional vector space. Define*

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

This is called the ***exterior algebra*** *of* V.

Definition 13.11. *Let* V *be a finite-dimensional vector space. For* $v \in V$*, we define* $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ *by*

$$i_v\omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1})$$

We have the alternate notation $i_v\omega = v \lrcorner\, \omega$*. This gives an operation* $\lrcorner : V \times \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$*, which is called* ***interior multiplication***.

Definition 13.12. *Let* M *be a smooth manifold, and let* $X \in \mathfrak{X}(M)$ *and* $\omega \in \Omega^k(M)$*. Then we define* $\boldsymbol{X \lrcorner\, \omega} \in \Omega^{k-1}(M)$ *(alternately* $i_X\omega$*) by*

$$(X \lrcorner\, \omega)_p = X_p \lrcorner\, \omega_p$$

Definition 13.13. *Let* $I = (i_1, \ldots, i_k)$*. We define*

$$\boldsymbol{dx^I} = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Definition 13.14. *Let* M *be a smooth manifold. We define*

$$\Lambda^k T^*M = \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$$

Sections of $\Lambda^k T^*M$ *are called* ***k-forms***. *That is, a k-form is a tensor field whose value at each point is an alternating tensor. In a smooth chart* (U, x^i)*, a k-form* ω *can be written as*

$$\omega = \sum_I{}' \omega_I dx^I$$

Definition 13.15. *The space of smooth k-forms is denoted* $\boldsymbol{\Omega^k(M)}$*. We then define*

$$\Omega^*(M) = \bigoplus_{k=1}^{N} \Omega^k(M)$$

With the wedge product, $\Omega^*(M)$ *is an associative, annticommtative, graded algebra.*

Definition 13.16. *Let* M *be a smooth manifold. The* ***exterior derivative*** *is the unique operator* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *such that* d *is linear over* $\mathbb{R}$*,* $d^2 = 0$*, and*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

for $\omega \in \Omega^k(M)$ *and* $\eta \in \Omega^l(M)$*, and for* $f \in C^\infty(M) = \Omega^0(M)$ *we have* $df(X) = Xf$*. In a smooth chart* (U, x^i) *we have*

$$d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^I$$

(summing over increasing multi-indices J*.)*

Definition 13.17. *A differential form* ω *is* ***closed*** *if* $d\omega = 0$*.*

Definition 13.18. *A differential form* ω *is* ***exact*** *if* $\omega = d\eta$ *for some other differential form* η*.*

14 Chapter 15 - Orientation

Definition 14.1. *Let* V *be an* n*-dimensional real vector space. We define an equivalence relation on the ordered bases of* V *by defining two bases to be equivalent if the transition matrix between them has positive determinant. An* ***orientation*** *on* V *is a choice of one of these two equivalence classes. Any basis in the chosen equivalence class is called* ***positively oriented****, and a basis in the other class is* ***negatively oriented****.*

Definition 14.2. *A* ***pointwise orientation*** *on a manifold* M *is a choice of orientation for each tangent space* T_pM*.*

Definition 14.3. *Let* M *be a smooth manifold and* U *an open subset. An* ***oriented local frame*** *on* U *is a local frame* $(E_1, \dots, E_n)$ *such that at each* $p \in U$ *the basis* $(E_1|_p, \dots, E_n|_p)$ *for* T_pM *is postively oriented.*

Definition 14.4. *A pointwise orientation on a manifold* M *is* ***continuous*** *if every* $p \in M$ *is contained in some oriented local frame.*

Definition 14.5. *An* ***orientation*** *for a manifold* M *is a continuous pointwise orientation.*

Definition 14.6. *A manifold is* ***orientable*** *if it has an orientation.*

Definition 14.7. *A smooth atlas for a manifold is* ***consistenly oriented*** *if the transition map between any two chart functions has positive Jacobian determinant everywhere on the intersection.*

Definition 14.8. *Let* $F : M \to N$ *be a local diffeomorphism of nonzero dimensional manifolds.* F *is* ***orientation preserving*** *if for each* $p \in M$ *the isomorphism* dF_p *maps positively oriented bases of* T_pM *to positively oriented bases of* $T_{F(p)}N$*.*

15 Chapter 16 - Integration on manifolds

Definition 15.1. *A **domain of integration** in $\mathbb{R}^n$ is a bounded subset whose boundary has measure zero.*

Definition 15.2. *Let $D \subset \mathbb{R}^n$ be a domain of integration and let $\omega = f dx^1 \wedge \ldots \wedge dx^n$ be an n-form on D, where $f : \overline{D} \to \mathbb{R}$ is continuous. The **integral of ω over D** is defined to be*

$$\int_D \omega = \int_D \omega \; dx^1 \wedge \ldots \wedge dx^n = \int_D f \; dx^1 \ldots dx^n$$

where the integral on the right is a Lebesgue integral.

Definition 15.3. *Let M be an oriented smooth n-manifold and let ω be an n-form on M. If ω is compactly supported in the domain of a positively oriented smooth chart (U, ϕ), then we define*

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega$$

Note that one can show that the integral does not depend on the choice of U.

Definition 15.4. *Let M be an oriented smooth n-manifold, and ω a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp} \omega$ by positively oriented charts, and $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we define the **integral of ω over M** to be*

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

where we compute each integral $\int_M \psi_i \omega$ using the previous definition. Note that one can show that the integral does not depend on the choice of open cover or the choice of partition of unity.

16 Chapter 17 - de Rham cohomology

Definition 16.1. *Let M be a smooth manifold and n a non-negative integer. The map $d : \Omega^n(M) \to \Omega^{n+1}(M)$ is linear, so its kernel and image are linear subspaces. We define*

$$Z^n(M) = \ker \big(d : \Omega^n(M) \to \Omega^{n+1}(M) \big) = \{\textit{closed } n\textit{-forms}\}$$
$$B^n(M) = \operatorname{im} \big(d : \Omega^{n-1}(M) \to \Omega^n(M) \big) = \{\textit{exact } n\textit{-forms}\}$$

Definition 16.2. *Since every exact form is closed, $B^n(M) \subset Z^n(M)$. Thus we can define the **nth de Rham cohomology group** to be the quotient space*

$$H^n(M) = \frac{Z^n(M)}{B^n(M)}$$

It's pretty dumb that it is called a group, because it is actually a vector space. A vector space is a group, but still. Oh well.

Definition 16.3. *Let ω be an n-form on a manifold M. Let $[\omega]$ be the equivalence class of ω in $H^n(M)$. If $[\omega] = [\omega']$, then ω and ω' are **cohomologous**. That is, $\omega - \omega'$ is exact.*

Definition 16.4. *A topological space is **contractible** if the identity map is homotopic to a constant map.*